

HEAT CONTENT AND SMALL TIME ASYMPTOTICS FOR SCHRÖDINGER OPERATORS ON \mathbb{R}^d

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ABSTRACT. This paper studies the *heat content* for Schrödinger operators of the fractional Laplacian $(-\Delta)^{\frac{\alpha}{2}}$, $0 < \alpha \leq 2$ in \mathbb{R}^d , $d \geq 1$. Employing probabilistic and analytic techniques, a small time asymptotic expansion formula is given and the “*heat content invariants*” are identified. These results are new even in the case of the Laplacian, $\alpha = 2$.

1. INTRODUCTION

Let $0 < \alpha \leq 2$ and consider $X = \{X_t\}_{t \geq 0}$ a rotationally invariant α -stable process whose transition densities $p_t^{(\alpha)}(x, y) = p_t^{(\alpha)}(x - y)$ are uniquely determined by their Fourier transform (characteristic function) and which are given by

$$(1.1) \quad e^{-t|\xi|^\alpha} = \mathbb{E}^0[e^{-i\xi \cdot X_t}] = \int_{\mathbb{R}^d} e^{-iy \cdot \xi} p_t^{(\alpha)}(y) dy,$$

for all $t > 0$, $\xi \in \mathbb{R}^d$, $d \geq 1$. Henceforth, \mathbb{E}^x will represent the expectation of the process starting at x .

For the purposes of this paper, we need to take into consideration both the spectral and integral definition for the infinitesimal generator associated to X , denoted here by $H_\alpha = (-\Delta)^{\frac{\alpha}{2}}$. In the spectral theoretic sense, $(-\Delta)^{\frac{\alpha}{2}}$ is a positive and self-adjoint linear operator with domain

$$\left\{ f \in L^2(\mathbb{R}^d) : |\xi|^\alpha \widehat{f}(\xi) \in L^2(\mathbb{R}^d) \right\}$$

satisfying

$$(1.2) \quad \widehat{(-\Delta)^{\frac{\alpha}{2}} f}(\xi) = |\xi|^\alpha \widehat{f}(\xi),$$

where \widehat{f} denotes the Fourier transform of f . Moreover, for $f \in \mathcal{S}(\mathbb{R}^d)$, where $\mathcal{S}(\mathbb{R}^d)$ is the set of rapidly decreasing smooth functions, we have

$$(-\Delta)^{\frac{\alpha}{2}} f(x) = \left. \frac{d}{dt} e^{-t(-\Delta)^{\frac{\alpha}{2}}} f(x) \right|_{t=0},$$

where

$$e^{-t(-\Delta)^{\frac{\alpha}{2}}} f(x) = \mathbb{E}^x[f(X_t)] = \int_{\mathbb{R}^d} p_t^{(\alpha)}(x, y) f(y) dy$$

is the heat semigroup generated by X . On the other hand, $(-\Delta)^{\frac{\alpha}{2}}$ can also be expressed in the integral form

$$(-\Delta)^{\frac{\alpha}{2}} f(x) = C_{d,\alpha} \int_{\mathbb{R}^d} \frac{f(x) - f(y)}{|x - y|^{d+\alpha}} dy,$$

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where $C_{d,\alpha} > 0$ is a normalizing constant and the integral is understood in the principal value sense. The last expression allows us to rewrite the Dirichlet form associated to $(-\Delta)^{\frac{\alpha}{2}}$ (see [15] for further details)

$$(1.3) \quad \mathcal{E}_\alpha(f) = \langle (-\Delta)^{\frac{\alpha}{2}} f, f \rangle = \int_{\mathbb{R}^d} (-\Delta)^{\frac{\alpha}{2}} f(x) f(x) dx$$

as

$$(1.4) \quad \mathcal{E}_\alpha(f) = \frac{C_{d,\alpha}}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|f(x) - f(y)|^2}{|x - y|^{d+\alpha}} dx dy.$$

Notice that when $\alpha = 2$, due to integration by parts, we have

$$(1.5) \quad \mathcal{E}_2(f) = \int_{\mathbb{R}^d} (-\Delta f)(x) f(x) dx = \int_{\mathbb{R}^d} |\nabla f(x)|^2 dx,$$

which is the classical Dirichlet form of the Laplacian.

Let $V \in L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$. The linear operator $H_V = (-\Delta)^{\frac{\alpha}{2}} + V$, known as the Fractional Schrödinger operator, is self-adjoint and defined similarly as the infinitesimal generator of the heat semigroup,

$$e^{-tH_V} f(x) = \mathbb{E}^x[e^{-\int_0^t V(X_s) ds} f(X_t)],$$

for $f \in \mathcal{S}(\mathbb{R}^d)$. The heat kernel of e^{-tH_V} is given by the Feymann-Kac formula (see [16], [18] and [25])

$$(1.6) \quad p_t^{H_V}(x, y) = p_t^{(\alpha)}(x, y) \mathbb{E}_{x,y}^t \left[e^{-\int_0^t V(X_s) ds} \right],$$

where $\mathbb{E}_{x,y}^t$ denotes the expectation with respect to the stable process (bridge) starting at x and conditioned to be at y at time t .

With H_α , H_V and their heat kernels properly introduced, we now proceed to consider the heat trace for Schrödinger operators, which is defined by

$$Tr(e^{-tH_V} - e^{-tH_\alpha}) = \int_{\mathbb{R}^d} (p_t^{H_V}(x, x) - p_t^{(\alpha)}(x, x)) dx.$$

We set

$$(1.7) \quad \mathcal{T}_V^{(\alpha)}(t) = \frac{Tr(e^{-tH_V} - e^{-tH_\alpha})}{p_t^{(\alpha)}(0)}.$$

The small time asymptotic expansion for the Schrödinger operator corresponding to the case $\alpha = 2$ (that is, the behavior of the quantity $\mathcal{T}_V^{(2)}(t)$ as $t \downarrow 0$) has been extensively studied in the literature for many years by many authors. One reason for this interest is its connections and applications to spectral and scattering theory. It is well known that there is an asymptotic expansion in powers of t and that the coefficients, known as the “*heat invariants*”, encode rich information on scattering poles and properties of the potential V . (See for example, van den Berg [10], McKean and Moerbeke [20] and Melrose [21, 22].) For this reason, there has been a great deal of interest in obtaining explicit expressions for the coefficients in the expansions under suitable (but general enough) assumptions on the potentials. Following these works, Sá Barreto and the second author proved the existence of an asymptotic expansion as $t \downarrow 0$ and gave a formula for the “*heat invariants*” in terms of quantities involving the Fourier transform of the potential $V \in \mathcal{S}(\mathbb{R}^d)$. This expansion allows for the computation of several coefficients and this in turn gives information on scattering poles for the potential V ; see [6, pp. 2162–2163] for details. Applications of the techniques in [6] for Schrödinger operators over

compact Riemannian manifolds are given in Donnelly [14]. In [5], Yildirim and the second author proved a second order expansion for (1.7) as $t \downarrow 0$ valid for all $0 < \alpha \leq 2$ by imposing a Hölder continuous condition on the potential V similar to that imposed in [10] for the case $\alpha = 2$. In [1], the first author combined the techniques in [6] with probabilistic techniques to derive a general small time asymptotic expansion for $\mathcal{T}_V^{(\alpha)}(t)$ valid for all $0 < \alpha \leq 2$. While this expansion is similar to that in [6] for $\alpha = 2$, the formula for the α -heat invariants for $0 < \alpha < 2$ contains some rather complicated probabilistic quantities that are quite difficult to compute. Nevertheless, the expansion in [1] permits the computation of several heat invariants for the general α 's as in the case of [6] for $\alpha = 2$.

While by no means complete (many questions concerning scattering theory remain completely open) our current understanding of trace asymptotics for the Schrödinger operator for the fractional Laplacians has greatly improved in recent years. This is in stark contrast to trace asymptotic for the Dirichlet fractional Laplacian in smooth bounded domains of \mathbb{R}^d where progress has been slow. If $D \subset \mathbb{R}^d$ is a domain of finite volume and $p_t^D(x, y)$ is the heat kernel for the Laplacian in D with Dirichlet boundary condition, then the following quantity

$$Z_D(t) = \int_D p_t^D(x, x) dx,$$

is known as the *heat trace* of the Dirichlet heat semigroup for the domain D . As in the case of the Schrödinger semigroups on \mathbb{R}^d , this quantity has been extensively investigated in the literature. In 1954, S. Minakshiusundaram [23] proved that if $D \subset \mathbb{R}^d$ is a bounded domain with smooth boundary (his result is for general manifolds with boundaries), then there are constants $c_j(D)$ such that for all $N \geq 3$,

$$(1.8) \quad Z_D(t) = \frac{1}{(4\pi t)^{d/2}} \left\{ |D| - \frac{\sqrt{\pi}}{2} |\partial D| t^{1/2} + \sum_{j=3}^N c_j(D) t^{j/2} + \mathcal{O}(t^{(N+1)/2}) \right\},$$

as $t \downarrow 0$. The *heat invariants* $c_j(D)$ have also been the source of intense interest for many years, specially following the foundational work of M. Kac [17] and McKean and Singer [19]. In [3] and [4], a second order expansion is computed for the Dirichlet fractional Laplacian valid for all $0 < \alpha \leq 2$. However, a general asymptotic expansion for the trace of fractional Laplacian similar to (1.8) remains an interesting open problem.

There are other spectral functions whose asymptotic expansions similarly encode important geometric information for the domain D . One of these is the *heat content* defined by

$$Q_D(t) = \int_D \int_D p_t^D(x, y) dy dx.$$

It represents the total amount of heat in the domain D by time t . Like the trace, it has an asymptotic expansion of the form

$$(1.9) \quad Q_D(t) = \sum_{j=0}^N a_j(D) t^{j/2} + \mathcal{O}(t^{(N+1)/2})$$

and the first few coefficients (called *heat content invariants*) have been calculated. (See van den Berg and Gilkey [8] and van den Berg, Gilkey, Kirsten and Kozlov [9] for more on the expansion and the calculation of coefficients.) The following result was proved by van den Berg and Le Gall in [12] for smooth domains $D \subset \mathbb{R}^d$, $d \geq 2$.

$$(1.10) \quad Q_D(t) = |D| - \frac{2}{\sqrt{\pi}} |\partial D| t^{1/2} + \left(2^{-1}(d-1) \int_{\partial D} H(s) ds \right) t + \mathcal{O}(t^{3/2}),$$

as $t \downarrow 0$. Here, $H(s)$ denotes the mean curvature at the point $s \in \partial D$. For more on the heat content asymptotics and its connections to the eigenvalues (spectrum) of the Laplacian in the domain D , we refer the reader to van den Berg, Dryden and Kappeler [7] and the many references to the literature contained therein.

Question 1.1. *Is there an expansion similar to (1.9) for the Dirichlet semigroup of stable processes for smooth bounded domains D , and can one compute the first few coefficients? In particular, is there a version of (1.10) for stable processes?*

As of now, these too remain challenging open questions. Even obtaining a second order asymptotic seems to be very challenging. We remark that the tools to show the existence of the asymptotic expansion (1.10) depend strongly on the fact that Brownian motion $B = \{B_t\}_{t \geq 0}$ on \mathbb{R}^d is obtained by taking d -independent copies of a 1-dimensional Brownian motion as its coordinates. This facilitates many calculations in the above expansions, often reducing matters to one dimensional problems; see for example [11]. Unfortunately, these type of arguments completely fail for stable processes. To understand more what these difficulties entail, we refer the reader to [3] and [4] where similar issues have to be confronted for trace asymptotics.

The above mentioned results on the trace of Schrödinger operators on \mathbb{R}^d and those for the Dirichlet semigroup motivate the study of what we will call “the heat content for Schrödinger semigroups” and which we define by

$$(1.11) \quad \begin{aligned} Q_V^{(\alpha)}(t) &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left\{ p_t^{H_V}(x, y) - p_t^{(\alpha)}(x, y) \right\} dx dy \\ &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} p_t^{(\alpha)}(x, y) \mathbb{E}_{x, y}^t \left[e^{-\int_0^t V(X_s) ds} - 1 \right] dx dy. \end{aligned}$$

Notice that the second equality comes from (1.6). To the best of our knowledge, this quantity has not been studied in the literature before even in the case of the Laplacian.

Before stating our results, we elaborate further on the name “heat content”. Recall that the heat kernels for the semigroups e^{-tH_α} and e^{-tH_V} of the operators $(-\Delta)^{\frac{\alpha}{2}}$ and $(-\Delta)^{\frac{\alpha}{2}} + V$, respectively, satisfy the heat equations

$$\frac{d}{dt} p_t^{(\alpha)}(x, y) = -(-\Delta)_x^{\frac{\alpha}{2}} p_t^{(\alpha)}(x, y), \quad t > 0, \quad (x, y) \in \mathbb{R}^{2d},$$

with initial condition

$$p_0^{(\alpha)}(x, y) = \delta(x - y),$$

and

$$\frac{d}{dt} p_t^{H_V}(x, y) = -[(-\Delta)_x^{\frac{\alpha}{2}} + V(x)] p_t^{H_V}(x, y), \quad t > 0, \quad (x, y) \in \mathbb{R}^{2d}$$

with

$$p_0^{H_V}(x, y) = \delta(x - y).$$

Consequently, the function

$$u(t, x, y) = p_t^{H_V}(x, y) - p_t^{(\alpha)}(x, y)$$

satisfies

$$\frac{d}{dt}u(t, x, y) = -(-\Delta)_x^{\alpha/2}u(t, x, y) - V(x)p_t^{H_V}(x, y), \quad (t, x, y) \in (0, +\infty) \times \mathbb{R}^{2d}$$

with initial condition

$$u(0, x, y) = 0, \quad (x, y) \in \mathbb{R}^{2d}.$$

From (1.6), we observe that when $V \leq 0$, we have $u(t, x, y) \geq 0$ so that we can interpret $u(t, x, y)$ as a temperature function that reflects the excess of heat generated by the potential V at time $t > 0$ at the point (x, y) . Similarly, when $V \geq 0$, $u(t, x, y) \leq 0$, which can be interpreted as a loss of heat. Likewise,

$$U(t, x) = \int_{\mathbb{R}^d} u(t, x, y) dy$$

can be regarded as a temperature function defined on \mathbb{R}^d which lets us interpret $Q_V^{(\alpha)}(t)$ as the amount of heat that the Euclidean space \mathbb{R}^d has gained, or lost, by time $t > 0$ with respect to the potential V . One of our goals is to compare the expansion as $t \downarrow 0$ for the heat content $Q_V^{(\alpha)}(t)$ to that of $\mathcal{T}_V^{(\alpha)}(t)$ proved in [6] ($\alpha = 2$ case), [5] and [1] ($0 < \alpha \leq 2$ case).

We proceed to state our main results. The first two theorems correspond to the results for the trace proved in [10] for $\alpha = 2$ and in [5] for $0 < \alpha < 2$. The first theorem provides the first term whereas the second theorem yields a second order expansions under the assumption of a Hölder continuity on the potential V . Both theorems provide uniform bounds for the remainder term for all positive times.

Theorem 1.1.

- (i) Assume $V \in L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$. Then if $V : \mathbb{R}^d \rightarrow (-\infty, 0]$, we have for all $t > 0$ that

$$-t \int_{\mathbb{R}^d} V(x) dx \leq Q_V^{(\alpha)}(t) \leq -t \int_{\mathbb{R}^d} V(x) dx \left(1 + \frac{1}{2} t \|V\|_\infty e^{t\|V\|_\infty} \right).$$

The last inequality implies that

$$Q_V^{(\alpha)}(t) = -t \int_{\mathbb{R}^d} V(x) dx + \mathcal{O}(t^2),$$

as $t \downarrow 0$.

- (ii) For $V \in L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$, we obtain for all $t > 0$

$$\left| Q_V^{(\alpha)}(t) + t \int_{\mathbb{R}^d} V(x) dx \right| \leq t^2 \|V\|_1 \|V\|_\infty e^{t\|V\|_\infty}.$$

In particular,

$$Q_V^{(\alpha)}(t) = -t \int_{\mathbb{R}^d} V(x) dx + \mathcal{O}(t^2),$$

as $t \downarrow 0$.

Theorem 1.2. Suppose $V \in L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$. Assume that V is also uniformly Hölder continuous of order γ . That is, there exists a positive constant M such that $|V(x) - V(y)| \leq M|x - y|^\gamma$, for all $x, y \in \mathbb{R}^d$, with $0 < \gamma < \min\{1, \alpha\}$, $0 < \alpha \leq 2$. Then, for all $t > 0$

$$\left| Q_V^{(\alpha)}(t) + t \int_{\mathbb{R}^d} V(x) dx - \frac{t^2}{2} \int_{\mathbb{R}^d} V^2(x) dx \right| \leq C(\gamma, \alpha) \|V\|_1 \left(\|V\|_\infty^2 e^{t\|V\|_\infty} t^3 + t^{\frac{\gamma}{\alpha}+2} \right).$$

In particular,

$$Q_V^{(\alpha)}(t) = -t \int_{\mathbb{R}^d} V(x) dx + \frac{t^2}{2} \int_{\mathbb{R}^d} V^2(x) dx + \mathcal{O}(t^{\frac{2}{\alpha}+2}),$$

as $t \downarrow 0$.

It is interesting to note here that in [5], it is shown that

$$\mathcal{T}_V^{(\alpha)}(t) = -t \int_{\mathbb{R}^d} V(\theta) d\theta + \frac{t^2}{2} \int_{\mathbb{R}^d} V^2(\theta) d\theta + \mathcal{O}(t^{\frac{2}{\alpha}+2}),$$

as $t \downarrow 0$ under the same conditions of Theorem 1.2. Thus, under the assumption of Hölder continuity we cannot distinguish between $Q_V^{(\alpha)}(t)$ and $\mathcal{T}_V^{(\alpha)}(t)$, as $t \downarrow 0$ as the scone order asymptotic expansion. In order to see the difference in these quantities for $t \downarrow 0$, we need to assume extra regularity conditions on V and go further in the expansion.

Our third result in this paper is a general asymptotic expansion in powers of t for potentials $V \in \mathcal{S}(\mathbb{R}^d)$ with an explicit form for the coefficients. In order to avoid the introduction of more complicated notation at this point, we postpone the result to Theorem 3.1 in §3. A special case of Theorem 3.1 where we can compute quite explicitly all the coefficients is the following theorem.

Theorem 1.3. *Let $V \in \mathcal{S}(\mathbb{R}^d)$ and $0 < \alpha \leq 2$. Then*

$$\begin{aligned} Q_V^{(\alpha)}(t) = & -t \int_{\mathbb{R}^d} V(\theta) d\theta + \frac{t^2}{2!} \int_{\mathbb{R}^d} V^2(\theta) d\theta - \frac{t^3}{3!} \left(\int_{\mathbb{R}^d} V^3(\theta) d\theta + \mathcal{E}_\alpha(V) \right) \\ & + \frac{t^4}{4!} \left(\int_{\mathbb{R}^d} V^4(\theta) d\theta + 2 \int_{\mathbb{R}^d} V^2(\theta) (-\Delta)^{\frac{\alpha}{2}} V(\theta) d\theta + \int_{\mathbb{R}^d} |(-\Delta)^{\frac{\alpha}{2}} V(\theta)|^2 d\theta \right) \\ & - \frac{t^5}{5!} \left(\int_{\mathbb{R}^d} V^5(\theta) d\theta + 2 \int_{\mathbb{R}^d} V^3(\theta) (-\Delta)^{\frac{\alpha}{2}} V(\theta) d\theta + 2 \int_{\mathbb{R}^d} V^2(\theta) (-\Delta)^{\frac{\alpha}{2}} V(\theta) d\theta \right. \\ & \left. + \int_{\mathbb{R}^d} V(\theta) |(-\Delta)^{\frac{\alpha}{2}} V(\theta)|^2 d\theta + \mathcal{E}_\alpha((-\Delta)^{\frac{\alpha}{2}} V) + \mathcal{E}_\alpha(V^2) \right) + \mathcal{O}(t^6), \end{aligned}$$

as $t \downarrow 0$. Here, \mathcal{E}_α is the Dirichlet form as defined in (1.4) and (1.5) whereas $(-\Delta)_2^{\frac{\alpha}{2}}$ is defined to be $(-\Delta)^{\frac{\alpha}{2}} \circ (-\Delta)^{\frac{\alpha}{2}}$.

The expansion above enables us to comment on the similarities and differences between the heat trace and the heat content. We start with the case $\alpha = 2$. It is proved in [6] that

$$\begin{aligned} \mathcal{T}_V^{(2)}(t) = & -t \int_{\mathbb{R}^d} V(\theta) d\theta + \frac{t^2}{2!} \int_{\mathbb{R}^d} V^2(\theta) d\theta - \frac{t^3}{3!} \left(\int_{\mathbb{R}^d} V^3(\theta) d\theta + \frac{1}{2} \mathcal{E}_2(V) \right) \\ & + \frac{t^4}{4!} \left(\int_{\mathbb{R}^d} V^4(\theta) d\theta + 2 \int_{\mathbb{R}^d} V(\theta) |\nabla V(\theta)|^2 d\theta + \frac{1}{5} \int_{\mathbb{R}^d} |(-\Delta) V(\theta)|^2 d\theta \right) \\ & - \frac{t^5}{5!} \int_{\mathbb{R}^d} \left(V^5(\theta) d\theta + \frac{3}{42} |\nabla(-\Delta) V(\theta)|^2 + 5 V^2(\theta) |\nabla V(\theta)|^2 \right. \\ & \left. + \frac{15}{27} V(\theta) |(-\Delta) V(\theta)|^2 + \frac{4}{9} V(\theta) \left(\sum_{i,j=1}^d \partial_{x_i} \partial_{x_j} V(\theta) \right)^2 \right) d\theta + \mathcal{O}(t^6), \end{aligned}$$

as $t \downarrow 0$.

From

$$\int_{\mathbb{R}^d} V^2(\theta)(-\Delta V)(\theta)d\theta = 2 \int_{\mathbb{R}^d} V(\theta)|\nabla V(\theta)|^2 d\theta$$

and

$$\int_{\mathbb{R}^d} |\nabla(-\Delta)V(\theta)|^2 d\theta = \int_{\mathbb{R}^d} (-\Delta V)(\theta)(-\Delta)_2 V(\theta)d\theta = \mathcal{E}_2((-\Delta)V)$$

and other similar identities, we note that by Theorem 1.3 the same integrands are involved in the expansion of both the heat trace and the heat content and both expansions behave similarly as $t \downarrow 0$. On the other hand, in the case $0 < \alpha < 2$, the heat trace and the heat content have completely different behavior for small t . In fact, the asymptotic expansion provided in [1] for $\mathcal{T}_V^{(\alpha)}(t)$ is dimensional dependent, unlike the situation of $\alpha = 2$. That is, the powers of t in the expansion depend on the location of α relative to the dimension d . One of the strongest result obtained in [1] is that

$$(1.12) \quad \begin{aligned} \mathcal{T}_V^{(\alpha)}(t) = & -t \int_{\mathbb{R}^d} V(\theta)d\theta + \frac{t^2}{2!} \int_{\mathbb{R}^d} V^2(\theta)d\theta - \frac{t^3}{3!} \int_{\mathbb{R}^d} V^3(\theta)d\theta \\ & - \mathcal{L}_{d,\alpha} t^{2+\frac{2}{\alpha}} \mathcal{E}_2(V) + \mathcal{O}(t^4), \end{aligned}$$

for all $d \geq 1$ and $\frac{3}{2} < \alpha < 2$ as $t \downarrow 0$, where $\mathcal{L}_{d,\alpha} > 0$ is a constant depending on the $\alpha/2$ -subordinator related to X . There, it is also shown that the same integrals involved in the expansion for $\mathcal{T}_V^{(2)}(t)$ appear in different positions in the corresponding expansion for $\mathcal{T}_V^{(\alpha)}(t)$ according to the given α under consideration. With these observations and Theorem 3.1 below, we conclude that the expansion of $Q_V^{(\alpha)}(t)$ gives information on the action of the operator $(-\Delta)^{\frac{\alpha}{2}}$ on the potential V and produces functions of t of the form ct^n , $n \in \mathbb{N}$, with explicit real numbers c . On the other hand, according to the results in [1], $\mathcal{T}_V^{(\alpha)}(t)$ gives information on the action of Δ on V and produces powers of t of the form $c_{\alpha,d} t^{n+\frac{j}{\alpha}}$, $n, j \in \mathbb{N}$, where $c_{\alpha,d}$, although explicitly given, are as of now quite difficult to compute for general d and α . For more on this, we refer the reader to [1].

The paper is organized as follows. In §2, we show that $Q_V^{(\alpha)}(t)$ is a well defined function for every bounded and integral potential V and prove a series of lemmas needed for the proof of Theorem 1.1 and Theorem 1.2. In §3, we prove the existence of a general expansion for $Q_V^{(\alpha)}(t)$ (Theorem 3.1) under the assumption that the potential is a rapidly decreasing smooth function. This is done using Fourier Transform techniques. Lastly, in §4, we compute the first five terms in the expansion obtained in §3 which proves Theorem 1.3.

2. PROOF OF THEOREMS 1.1 AND 1.2.

We start this section by proving that $Q_V^{(\alpha)}(t)$ given by (1.11) is a well-defined function for all $t \geq 0$ as long as V is bounded and integrable. We begin by observing that the elementary inequality $|e^z - 1| \leq |z| e^{|z|}$ gives

$$\left| Q_V^{(\alpha)}(t) \right| \leq e^{t\|V\|_\infty} \int_{\mathbb{R}^{2d}} p_t^{(\alpha)}(x, y) \mathbb{E}_{x,y}^t \left[\int_0^t |V(X_s)| ds \right] dx dy.$$

Next, by Fubini's theorem and the properties of stable bridge (see [5], [13] and (2.6) below) the integral term in the right hand side of the above inequality equals

$$(2.1) \quad \int_{\mathbb{R}^{2d}} p_t^{(\alpha)}(x, y) \left(\int_0^t \int_{\mathbb{R}^d} |V(z)| \frac{p_{t-s}^{(\alpha)}(x, z) p_s^{(\alpha)}(z, y)}{p_t^{(\alpha)}(x, y)} dz ds \right) dx dy = \\ \int_0^t \int_{\mathbb{R}^d} |V(z)| \left(\int_{\mathbb{R}^d} p_{t-s}^{(\alpha)}(x, z) dx \int_{\mathbb{R}^d} p_s^{(\alpha)}(z, y) dy \right) dz ds = t \|V\|_1,$$

where we have used the well known facts that for all $x, z \in \mathbb{R}^d$ and $t > 0$, $p_t^{(\alpha)}(x, z) = p_t^{(\alpha)}(z, x)$ and $\int_{\mathbb{R}^d} p_t^{(\alpha)}(x, z) dx = 1$. Thus, we conclude that the heat content satisfies

$$|Q_V^{(\alpha)}(t)| \leq t \|V\|_1 e^{t \|V\|_\infty}.$$

Therefore, $Q_V^{(\alpha)}(t)$ is well-defined for all $t \geq 0$ and bounded on any interval $(0, T]$, $T > 0$, provided $V \in L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$. It is also worth noting here that the previous argument together with Taylor's expansion of the exponential function (see (3.5) below) show that

$$(2.2) \quad Q_V^{(\alpha)}(t) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \int_{\mathbb{R}^{2d}} p_t^{(\alpha)}(x, y) \mathbb{E}_{x,y}^t \left[\left(\int_0^t V(X_s) ds \right)^k \right] dx dy,$$

where the sum is absolutely convergent for all $t > 0$.

It is advantageous at this point to give a different expression for the equation (2.2) in terms of the stable bridge in order to obtain further formulas for the coefficients and estimates for the remainders in the forthcoming sections. Before proceeding, we introduce some notation to conveniently express our formulas below. For $k \in \mathbb{N}$, we set

$$(2.3) \quad I_k = \left\{ \lambda^{(k)} = (\lambda_k, \lambda_{k-1}, \dots, \lambda_1) \in [0, 1]^k : 0 < \lambda_k < \lambda_{k-1} < \dots < \lambda_1 < 1 \right\},$$

$$d\lambda^{(k)} = d\lambda_k d\lambda_{k-1} \dots d\lambda_1, \quad z^{(k)} = (z_1, \dots, z_k) \in \mathbb{R}^{kd}, \quad dz^{(k)} = dz_k \dots dz_1,$$

$$V_k(z^{(k)}) = V_k(z_1, \dots, z_k) = \prod_{i=1}^k V(z_i), \quad p(t, z^{(k)}) = \prod_{j=1}^{k-1} p_{t(\lambda_j - \lambda_{j+1})}^{(\alpha)}(z_j, z_{j+1}).$$

It is well known [26] that

$$(2.4) \quad \left(\int_0^1 \tilde{V}(s) ds \right)^k = k! \int_{I_k} \prod_{i=1}^k \tilde{V}(\lambda_i) d\lambda^{(k)},$$

for any $\tilde{V} : [0, 1] \rightarrow \mathbb{R}$ integrable.

Lemma 2.1. *For any $t > 0$ and $J \geq 2$,*

$$Q_V^{(\alpha)}(t) = -t \int_{\mathbb{R}^d} V(\theta) d\theta + \sum_{k=2}^J (-t)^k \int_{I_k} \int_{\mathbb{R}^{kd}} V_k(z^{(k)}) p(t, z^{(k)}) dz^{(k)} d\lambda^{(k)} + R_{J+1}(t),$$

where

$$(2.5) \quad |R_{J+1}(t)| \leq t^{J+1} \|V\|_1 \|V\|_\infty^J e^{t \|V\|_\infty}.$$

Proof. Set

$$R_{J+1}(t) = \sum_{k=J+1}^{\infty} \frac{(-1)^k}{k!} \int_{\mathbb{R}^{2d}} p_t^{(\alpha)}(x, y) \mathbb{E}_{x,y}^t \left[\left(\int_0^t V(X_s) ds \right)^k \right] dx dy.$$

It is clear by (2.1) that

$$\begin{aligned} |R_{J+1}(t)| &\leq \sum_{k=J+1}^{\infty} \frac{(t\|V\|_{\infty})^{k-1}}{k!} \int_{\mathbb{R}^{2d}} p_t^{(\alpha)}(x, y) \mathbb{E}_{x,y}^t \left[\int_0^t |V(X_s)| ds \right] dx dy \\ &\leq t\|V\|_1 \sum_{k=J+1}^{\infty} \frac{(t\|V\|_{\infty})^{k-1}}{k!} \leq t^{J+1} \|V\|_1 \|V\|_{\infty}^J e^{t\|V\|_{\infty}}. \end{aligned}$$

On the other hand, by making a suitable change of variables and appealing to (2.4) we observe that

$$\begin{aligned} \mathbb{E}_{x,y}^t \left[\left(\int_0^t V(X_s) ds \right)^k \right] &= t^k \mathbb{E}_{x,y}^t \left[\left(\int_0^1 V(X_{ts}) ds \right)^k \right] \\ &= k! t^k \mathbb{E}_{x,y}^t \left[\int_{I_k} V(X_{t\lambda_1}) \dots V(X_{t\lambda_k}) d\lambda^{(k)} \right]. \end{aligned}$$

We recall that the finite dimensional distributions of the stable bridge (see [5], [13] and references therein for details) are given by

$$\begin{aligned} (2.6) \quad \mathbb{P}_{x,y}^t (X_{t\lambda_1} \in dz_1, X_{t\lambda_2} \in dz_2, \dots, X_{t\lambda_k} \in dz_k) \\ = \frac{1}{p_t^{(\alpha)}(x, y)} \prod_{j=0}^k p_{t(\lambda_j - \lambda_{j+1})}^{(\alpha)}(z_j, z_{j+1}) dz^{(k)}, \end{aligned}$$

where $z_0 = x$, $z_{k+1} = y$, $\lambda_0 = 1$, and $\lambda_{k+1} = 0$. Hence, using the fact that

$$\int_{\mathbb{R}^d} p_{t(1-\lambda_1)}^{(\alpha)}(x, z_1) dx = \int_{\mathbb{R}^d} p_{t\lambda_k}^{(\alpha)}(z_k, y) dy = 1,$$

the notation given in (2.3) and the finite distribution for the stable bridge given above, we conclude by Fubini's theorem that

$$\begin{aligned} (2.7) \quad \int_{\mathbb{R}^{2d}} p_t^{(\alpha)}(x, y) \mathbb{E}_{x,y}^t \left[\left(\int_0^t V(X_s) ds \right)^k \right] dx dy \\ = k! t^k \int_{I_k} \int_{\mathbb{R}^{kd}} \prod_{i=1}^k V(z_i) \prod_{j=1}^{k-1} p_{t(\lambda_j - \lambda_{j+1})}^{(\alpha)}(z_j, z_{j+1}) dz^{(k)} d\lambda^{(k)} \\ = k! t^k \int_{I_k} \int_{\mathbb{R}^{kd}} V_k(z^{(k)}) p(t, z^{(k)}) dz^{(k)} d\lambda^{(k)}. \end{aligned}$$

Therefore, the lemma follows from equation (2.2). \square

Proof of Theorem 1.1: Setting $a = \int_0^t V(X_s) ds$ and $b = t\|V\|_{\infty}$, we observe that $-b \leq a < 0$. We use the elementary inequality

$$-a \leq e^{-a} - 1 \leq -a \left(1 + \frac{1}{2} b e^b \right),$$

to obtain

$$(2.8) \quad -\int_0^t V(X_s)ds \leq e^{-\int_0^t V(X_s)ds} - 1 \leq -\int_0^t V(X_s)ds \left(1 + \frac{1}{2}t\|V\|_\infty e^{t\|V\|_\infty}\right).$$

By taking expectations $\mathbb{E}_{x,y}^t$ at both sides of (2.8), multiplying through by $p_t^{(\alpha)}(x,y)$, integrating on \mathbb{R}^{2d} with respect to x and y and appealing to (2.1) where $|V|$ is replaced by $-V \geq 0$, we arrive at

$$-t \int_{\mathbb{R}^d} V(x)dx \leq Q_V^{(\alpha)}(t) \leq -t \int_{\mathbb{R}^d} V(x)dx \left(1 + \frac{1}{2}t\|V\|_\infty e^{t\|V\|_\infty}\right).$$

Thus, (i) follows.

By using (2.1) and (2.2), we have

$$\begin{aligned} \left| Q_V^{(\alpha)}(t) + t \int_{\mathbb{R}^d} V(x)dx \right| &\leq \sum_{k=2}^{\infty} \frac{1}{k!} \int_{\mathbb{R}^{2d}} p_t^{(\alpha)}(x,y) \mathbb{E}_{x,y}^t \left[\left(\int_0^t |V(X_s)| ds \right)^k \right] dx dy \\ &\leq t\|V\|_1 \sum_{k=2}^{\infty} \frac{1}{k!} t^{k-1} \|V\|_\infty^{k-1} \leq t^2 \|V\|_1 \|V\|_\infty e^{t\|V\|_\infty}, \end{aligned}$$

which in turn proves (ii).

Proof of Theorem 1.2: We start by recalling two basic facts about the α -stable process X , $0 < \alpha \leq 2$. First,

$$(2.9) \quad \mathbb{E}^0 [|X_1|^\gamma] < \infty,$$

whenever $\gamma < \alpha < 2$. As for $\alpha = 2$, the above fact is also true, since in this case $\gamma = 1$. Secondly,

$$(2.10) \quad X_t = t^{1/\alpha} X_1,$$

in law as we can see from the characteristic function (1.1).

The Hölder continuity assumption on V , as we shall see in the next lemma, enables us to estimate the second term in (2.2).

Lemma 2.2. *Under the same assumptions on the potential V given in Theorem 1.2, we have for all $t > 0$ that*

$$\int_{\mathbb{R}^{2d}} p_t^{(\alpha)}(x,y) \mathbb{E}_{x,y}^t \left[\left(\int_0^t V(X_s)ds \right) \right] dx dy = 2t^2 \int_{\mathbb{R}^d} |V(x)|^2 dx + R(t),$$

where the remainder $R(t)$ satisfies

$$|R(t)| \leq C_0(\alpha, \gamma) \|V\|_1 t^{\frac{\gamma}{\alpha} + 2}.$$

Proof. We start by applying (2.7) with $k = 2$, so that

$$\begin{aligned} \int_{\mathbb{R}^{2d}} p_t^{(\alpha)}(x,y) \mathbb{E}_{x,y}^t \left[\left(\int_0^t V(X_s)ds \right)^2 \right] dx dy = \\ 2t^2 \int_0^1 \int_0^{\lambda_1} \int_{\mathbb{R}^{2d}} V(z_1)V(z_2) p_{t(\lambda_1 - \lambda_2)}^{(\alpha)}(z_2, z_1) dz_2 dz_1 d\lambda_2 d\lambda_1. \end{aligned}$$

Now,

$$(2.11) \quad \begin{aligned} & \int_{\mathbb{R}^{2d}} V(z_1) V(z_2) p_{t(\lambda_1 - \lambda_2)}^{(\alpha)}(z_2, z_1) dz_1 dz_2 = \\ & \int_{\mathbb{R}^{2d}} (V(z_1) - V(z_2)) V(z_2) p_{t(\lambda_1 - \lambda_2)}^{(\alpha)}(z_2, z_1) dz_1 dz_2 \\ & + \int_{\mathbb{R}^{2d}} V^2(z_2) p_{t(\lambda_1 - \lambda_2)}^{(\alpha)}(z_2, z_1) dz_1 dz_2. \end{aligned}$$

The second term on the right hand side of equality (2.11) equals $\int_{\mathbb{R}^d} V^2(x) dx$, whereas for the first term, by using (2.9), (2.10) and the Hölder continuity assumption on V , we have

$$(2.12) \quad \begin{aligned} & \left| \int_{\mathbb{R}^{2d}} (V(z_1) - V(z_2)) V(z_2) p_{t(\lambda_1 - \lambda_2)}^{(\alpha)}(z_2, z_1) dz_1 dz_2 \right| \\ & \leq M \int_{\mathbb{R}^{2d}} |z_1 - z_2|^\gamma |V(z_2)| p_{t(\lambda_1 - \lambda_2)}^{(\alpha)}(z_2, z_1) dz_1 dz_2 \\ & = M \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |z_1 - z_2|^\gamma p_{t(\lambda_1 - \lambda_2)}^{(\alpha)}(z_2, z_1) dz_1 \right) |V(z_2)| dz_2 \\ & = M \|V\|_1 \mathbb{E}^0[|X_{t(\lambda_1 - \lambda_2)}|^\gamma] \\ & = M \|V\|_1 (t(\lambda_1 - \lambda_2))^{\frac{\gamma}{\alpha}} \mathbb{E}^0[|X_1|^\gamma]. \end{aligned}$$

Thus, by using the fact that

$$\int_0^1 \int_0^{\lambda_1} (\lambda_1 - \lambda_2)^{\gamma/\alpha} d\lambda_2 d\lambda_1 = \left(\frac{\gamma}{\alpha} + 2\right)^{-1} \left(\frac{\gamma}{\alpha} + 1\right)^{-1},$$

we obtain that the conclusion of the lemma follows from (2.12), (2.11) by setting

$$R(t) = 2t^2 \int_0^1 \int_0^{\lambda_1} \int_{\mathbb{R}^{2d}} (V(z_1) - V(z_2)) V(z_2) p_{t(\lambda_1 - \lambda_2)}^{(\alpha)}(z_2, z_1) dz_1 dz_2 d\lambda_1 d\lambda_2.$$

□

Therefore, using that $t^3 \leq t^{2+\frac{\gamma}{\alpha}}$ for $t \in (0, 1)$, we have that Theorem 1.2 is a consequence of applying Lemma 2.2 to the following expression obtained in Lemma 2.1 when $J = 2$,

$$Q_V^{(\alpha)}(t) = -t \int_{\mathbb{R}^d} V(x) dx + \frac{1}{2} \int_{\mathbb{R}^{2d}} p_t^{(\alpha)}(x, y) E_{x, y}^t \left[\left(\int_0^t V(X_s) ds \right)^2 \right] dx dy + R_3(t),$$

where we already know that

$$|R_3(t)| \leq t^3 \|V\|_1 \|V\|_\infty^2 e^{t\|V\|_\infty}.$$

3. GENERAL EXPANSION FOR RAPIDLY DECREASING SMOOTH POTENTIAL

We have already seen in the previous section that by adding an extra regularity condition on the potential V , namely, Hölder continuity and using $X_t = t^{1/\alpha} X_1$ in law, we have been able to extract a second term in the expansion of $Q_V^{(\alpha)}(t)$. In this section, we will obtain more terms and find explicit expressions for these which as before will depend on the potential V .

Let $V \in \mathcal{S}(\mathbb{R}^d)$. We denote by \widehat{V} the Fourier transform of V with the normalization

$$(3.1) \quad \widehat{V}(\xi) = \int_{\mathbb{R}^d} e^{-ix \cdot \xi} V(x) dx.$$

We note that due to our definition of \widehat{V} , we have, by setting $\bar{d}\xi = (2\pi)^{-d} d\xi$,

(i) (Fourier inversion formula)

$$V(x) = \int_{\mathbb{R}^d} e^{ix \cdot \xi} \widehat{V}(\xi) \bar{d}\xi,$$

and

(ii) (Plancherel identity) For $f, g \in \mathcal{S}(\mathbb{R}^d)$,

$$\int_{\mathbb{R}^d} e^{-ix \cdot \xi} f(x) g(x) dx = \int_{\mathbb{R}^d} \widehat{f}(\theta) \widehat{g}(\xi - \theta) \bar{d}\theta.$$

The fact that $V \in \mathcal{S}(\mathbb{R}^d)$ will allow us to apply the inversion formula to each summand in Lemma 2.1 which in turn will provide the terms obtained in Theorem 1.3. To do this, we need the following proposition.

Proposition 3.1. *For any $k \geq 2$,*

$$(3.2) \quad \int_{\mathbb{R}^{kd}} \prod_{i=1}^k V(z_i) \prod_{j=1}^{k-1} p_{t(\lambda_j - \lambda_{j+1})}^{(\alpha)}(z_j, z_{j+1}) dz_k \dots dz_1 = \\ \int_{\mathbb{R}^{(k-1)d}} \widehat{V}\left(-\sum_{i=1}^{k-1} \theta_i\right) \prod_{i=1}^{k-1} \widehat{V}(\theta_i) \exp\left(-t \sum_{r=1}^{k-1} (\lambda_r - \lambda_{r+1}) \left|\sum_{m=1}^r \theta_m\right|^\alpha\right) \bar{d}\theta_{k-1} \dots \bar{d}\theta_1.$$

Proof. Under the notation given in (2.3) we have

$$V_k(z^{(k)}) = V_k(z_1, \dots, z_k) = \prod_{i=1}^k V(z_i); \\ p(t, z^{(k)}) = \prod_{r=1}^{k-1} p_{t(\lambda_r - \lambda_{r+1})}^{(\alpha)}(z_r, z_{r+1}).$$

By applying Fourier transform in \mathbb{R}^{kd} and Plancherel identity, we obtain

$$(3.3) \quad \int_{\mathbb{R}^{kd}} V_k(z^{(k)}) p(t, z^{(k)}) dz^{(k)} = \int_{\mathbb{R}^{kd}} \widehat{V}_k(\theta^{(k)}) \widehat{p}(t, -\theta^{(k)}) \bar{d}\theta^{(k)}.$$

Next, it follows easily that if $\theta^{(k)} = (\theta_1, \dots, \theta_k)$, $\theta_i \in \mathbb{R}^d$, then

$$\widehat{V}_k(\theta^{(k)}) = \prod_{i=1}^k \widehat{V}(\theta_i).$$

On the other hand, we claim that

$$(3.4) \quad \widehat{p}(t, -\theta^{(k)}) = (2\pi)^d \delta\left(\sum_{i=1}^k \theta_i\right) \exp\left(-t \sum_{r=1}^{k-1} (\lambda_r - \lambda_{r+1}) \left|\sum_{m=1}^r \theta_m\right|^\alpha\right).$$

To see this, we observe by (3.1) that

$$\widehat{p}(t, -\theta^{(k)}) = \int_{\mathbb{R}^{kd}} \exp \left(i \sum_{j=1}^k \xi_j \cdot \theta_j \right) \prod_{r=1}^{k-1} p_{t(\lambda_r - \lambda_{r+1})}^{(\alpha)}(\xi_r - \xi_{r+1}) d\xi^{(k)}.$$

By considering the substitutions $z_r = \xi_r - \xi_{r+1}$, $r \in \{1, \dots, k-1\}$, we have for any $j \in \{1, \dots, k-1\}$ that

$$\xi_j = \xi_k + \sum_{r=j}^{k-1} z_r.$$

Therefore, we obtain after interchanging the order of summation that

$$\sum_{j=1}^{k-1} \xi_j \cdot \theta_j = \sum_{r=1}^{k-1} z_r \cdot \left(\sum_{m=1}^r \theta_m \right) + \xi_k \cdot \sum_{i=1}^{k-1} \theta_i.$$

Thus, (3.4) follows by using that

$$\widehat{p_t^{(\alpha)}}(\xi) = e^{-t|\xi|^\alpha},$$

$$\int_{\mathbb{R}^d} \exp \left(i \xi_k \cdot \sum_{i=1}^k \theta_i \right) d\xi_k = (2\pi)^d \delta \left(\sum_{i=1}^k \theta_i \right),$$

and

$$\begin{aligned} \widehat{p}(t, -\theta^{(k)}) &= \int_{\mathbb{R}^{kd}} \exp \left(i \sum_{r=1}^{k-1} z_r \cdot \left(\sum_{m=1}^r \theta_m \right) + i \xi_k \cdot \sum_{i=1}^k \theta_i \right) \\ &\quad \times \prod_{r=1}^{k-1} p_{t(\lambda_r - \lambda_{r+1})}^{(\alpha)}(z_r) dz^{(k-1)} d\xi_k. \end{aligned}$$

Consequently, the conclusion of the proposition follows from (3.3) and (3.4). \square

We next recall the Taylor expansion for the exponential function

$$(3.5) \quad e^{-x} = \sum_{n=0}^M \frac{(-1)^n}{n!} x^n + \frac{(-1)^{M+1}}{(M+1)!} x^{M+1} e^{-x\beta_{M+1}(x)},$$

valid for every $x \geq 0$ and integer $M \geq 0$, where we call $\beta_{M+1}(x) \in (0, 1)$ the remainder of order $M+1$.

We also recall that for $k \geq 2$ integer, the Binomial theorem asserts that

$$(x_1 + x_2 + \dots + x_{k-1})^n = \sum_{\substack{(\ell_1, \dots, \ell_{k-1}) \in \mathbb{N}^{k-1}, \\ \ell_1 + \ell_2 + \dots + \ell_{k-1} = n}} \binom{n}{\ell_1, \ell_2, \dots, \ell_{k-1}} x_1^{\ell_1} x_2^{\ell_2} \dots x_{k-1}^{\ell_{k-1}}.$$

Next, bearing in mind the notation given in (2.3), we set $\gamma_r = \sum_{m=1}^r \theta_m$,

$$\ell^{(k-1)} = (\ell_1, \dots, \ell_{k-1}) \in \mathbb{N}^{k-1},$$

$$A(n, \ell^{(k-1)}) = \binom{n}{\ell_1, \ell_2, \dots, \ell_{k-1}} \int_{I_k} \prod_{i=1}^{k-1} (\lambda_i - \lambda_{i+1})^{\ell_i} d\lambda^{(k)},$$

and

$$(3.6) \quad T_k(t) = \int_{I_k} \int_{\mathbb{R}^{(k-1)d}} \prod_{i=1}^{k-1} \widehat{V}(\theta_i) \widehat{V}\left(-\sum_{i=1}^{k-1} \theta_i\right) \\ \times \exp\left(-t \sum_{r=1}^{k-1} (\lambda_r - \lambda_{r+1}) \left|\sum_{m=1}^r \theta_m\right|^\alpha\right) \bar{d}\theta^{(k-1)} d\lambda^{(k)}.$$

Therefore, under this notation, we obtain the following expansion for the term $T_k(t)$.

Corollary 3.1. *Let $M \geq 0$ and $k \geq 2$ be integers. Then*

$$T_k(t) = \sum_{n=0}^M \frac{(-t)^n}{n!} C_{n,k}(V) + R_{M+1}^{(k)}(t),$$

where

$$R_{M+1}^{(k)}(t) = \frac{(-t)^{M+1}}{(M+1)!} \int_{I_k} \int_{\mathbb{R}^{(k-1)d}} \prod_{i=1}^{k-1} \widehat{V}(\theta_i) \widehat{V}\left(-\sum_{i=1}^{k-1} \theta_i\right) \\ \times \left(\sum_{r=1}^{k-1} (\lambda_r - \lambda_{r+1}) |\gamma_r|^\alpha\right)^{M+1} e^{-\Upsilon} \bar{d}\theta^{(k-1)} d\lambda^{(k)},$$

for some nonnegative function $\Upsilon = \Upsilon(t, \lambda^{(k)}, \theta^{(k-1)}, M+1)$. The remainder satisfies

$$(3.7) \quad R_{M+1}^{(k)}(t) = \mathcal{O}(t^{M+1}),$$

as $t \downarrow 0$. Moreover, the coefficients are given by

$$C_{n,k}(V) = \sum_{\substack{(\ell_1, \dots, \ell_{k-1}) \in \mathbb{N}^{k-1}, \\ \ell_1 + \ell_2 + \dots + \ell_{k-1} = n}} A(n, \ell^{(k-1)}) \int_{\mathbb{R}^{(k-1)d}} \widehat{V}\left(-\sum_{i=1}^{k-1} \theta_i\right) \prod_{i=1}^{k-1} \widehat{V}(\theta_i) \\ \times \left|\sum_{m=1}^i \theta_m\right|^{\alpha \ell_i} \bar{d}\theta^{(k-1)}.$$

Proof. The formula for the coefficients is obtained by applying the Taylor expansion for the exponential function and the Binomial theorem to our expression of $T_k(t)$ in (3.6).

Next, we proceed to show our claim about the remainder. In order to do so, we point out that $V \in \mathcal{S}(\mathbb{R}^d)$ implies that all quantities to appear below are finite. Also, the constant C will depend on k, M and α and its value may change from line to line. It is easy to observe that for some $C > 0$, we have

$$\left(\sum_{r=1}^{k-1} (\lambda_r - \lambda_{r+1}) |\gamma_r|^\alpha\right)^{M+1} \leq C \max_{m=1, \dots, k-1} |\theta_m|^{\alpha(M+1)}.$$

In particular, if we let $\Lambda_r = \left\{ \theta^{(k-1)} \in \mathbb{R}^{d(k-1)} : \max_{m=1, \dots, k-1} |\theta_m| = |\theta_r| \right\}$, we arrive at

$$\begin{aligned} \left| R_{M+1}^{(k)}(t) \right| &\leq C t^{M+1} \sum_{r=1}^{k-1} \int_{\Lambda_r} |\widehat{V}|(-\gamma_{k-1}) |\widehat{V}(\theta_r)| |\theta_r|^{\alpha(M+1)} \prod_{i=1, i \neq r}^{k-1} |\widehat{V}|(\theta_i) d\theta^{(k-1)} \\ &\leq C t^{M+1} \|\widehat{V}\|_\infty \|(-\Delta)_{M+1}^{\frac{\alpha}{2}}(V)\|_1 \|\widehat{V}\|_1^{k-2}. \end{aligned}$$

Here, $(-\Delta)_{M+1}^{\frac{\alpha}{2}}$ stands for the composite of $(-\Delta)^{\frac{\alpha}{2}}$ with itself $M+1$ -times and this completes the proof. \square

With Corollary 3.1 at hand, we carry on showing the existence of a general expansion for $Q_V^{(\alpha)}(t)$ for small time.

Theorem 3.1. *For any integer $N \geq 2$,*

$$(3.8) \quad Q_V^{(\alpha)}(t) = -t \int_{\mathbb{R}^d} V(\theta) d\theta + \sum_{\ell=2}^N (-t)^\ell C_\ell(V) + \mathcal{O}(t^{N+1}),$$

as $t \downarrow 0$. Here,

$$C_\ell(V) = \sum_{\substack{n+k=\ell \\ 2 \leq k}} \frac{1}{n!} C_{n,k}(V),$$

where $C_{n,k}(V)$ as defined in Corollary 3.1.

Proof. As a result of Lemma 2.1, Corollary 3.1 and (3.2), we have for any integers $J \geq 2$ and $M \geq 0$ that

$$(3.9) \quad Q_V^{(\alpha)}(t) = -t \int_{\mathbb{R}^d} V(\theta) d\theta + \sum_{k=2}^J \sum_{n=0}^M \frac{(-t)^{k+n}}{n!} C_{n,k}(V) + R_{M+1, J+1}(t),$$

where

$$R_{M+1, J+1}(t) = R_{J+1}(t) + \sum_{k=2}^J (-t)^k R_{M+1}^{(k)}.$$

In other words, $R_{J+1, M+1}(t)$ is the sum of all those remainders provided by Lemma 2.1 and Corollary 3.1. We also point out that due to (2.5) and (3.7), we conclude

$$R_{M+1, J+1}(t) = \mathcal{O}\left(t^{\min\{J+1, M+3\}}\right),$$

as $t \downarrow 0$.

Since M and J are arbitrary, given $N \geq 2$, we may choose M and J as large as we desire so that

$$\min\{J+1, M+3\} \geq N+1$$

and such that formula (3.9) can be decomposed as follows

$$(3.10) \quad Q_V^{(\alpha)}(t) = -t \int_{\mathbb{R}^d} V(\theta) d\theta + \sum_{\substack{2 \leq n+k \leq N \\ 2 \leq k}} \frac{(-t)^{k+n}}{n!} C_{n,k}(V) + \tilde{R}_{N+1}(t),$$

where $\tilde{R}_{N+1}(t)$ is defined to be

$$\sum_{\substack{n+k \geq N+1 \\ 2 \leq k}} \frac{(-t)^{k+n}}{n!} C_{n,k}(V) + R_{M+1,J+1}(t).$$

Thus, it is easy to observe that $\tilde{R}_{N+1}(t) = \mathcal{O}(t^{N+1})$ as $t \downarrow 0$.

The conclusion of the theorem follows by noticing that the second terms on the right hand side of both (3.10) and (3.8) are the same under our definition of $C_\ell(V)$. \square

Before proceeding, we give an application concerning the coefficients $C_\ell(V)$. The corollary roughly says that we can characterize the potential V from the coefficients under some extra assumptions. This corollary should be compared to the result for the trace ($\alpha = 2$ case) given in [6, Corollary 2.1].

Corollary 3.2. *Let $V \in \mathcal{S}(\mathbb{R}^d)$ be such that $\hat{V} \geq 0$. If $C_\ell(V) = 0$ for some $\ell \geq 2$, then we must have $V(x) = 0$ for all $x \in \mathbb{R}^d$.*

Proof. By Theorem 3.1 and corollary 3.1, we see that $C_\ell(V) \geq 0$ for all $\ell \geq 2$ when $\hat{V} \geq 0$. In particular, the condition $C_\ell(V) = 0$ for some $\ell \geq 2$ implies that

$$C_{\ell-2,2}(V) = \binom{\ell-2}{\ell-2} \int_{I_2} (\lambda_1 - \lambda_2)^{\ell-2} d\lambda^{(2)} \int_{\mathbb{R}^d} \hat{V}(-\theta_1) \hat{V}(\theta_1) |\theta_1|^{\alpha(\ell-2)} d\theta_1 = 0.$$

Therefore, we must have $\hat{V}(-\theta_1) \hat{V}(\theta_1) = 0$ for all $\theta_1 \in \mathbb{R}^d$. Now by applying Plancherel identity we have

$$\int_{\mathbb{R}^d} |V(x)|^2 dx = \int_{\mathbb{R}^d} \hat{V}(-\theta_1) \hat{V}(\theta_1) d\theta_1 = 0$$

and this gives the claimed result. \square

4. COMPUTATION OF COEFFICIENTS

In this section we write down explicitly the first five coefficients of the asymptotic expansion given in (3.8). This also proves Theorem 1.3. All the results in the previous section also hold for $\alpha = 2$. Therefore we will consider $0 < \alpha \leq 2$.

In order to find the coefficients $C_3(V)$, $C_4(V)$ and $C_5(V)$, we will resort to Lemma 4.1 below. We start by observing that by means of the inversion formula, it follows easily that

$$(4.1) \quad C_{0,k}(V) = \frac{1}{k!} \int_{\mathbb{R}^d} V^k(\theta) d\theta,$$

for any integer $k \geq 2$.

Lemma 4.1. *Let $k \geq 2$ be an integer. Assume that $\{\ell_i, i \in \{1, \dots, k-1\}\}$ is a sequence of nonnegative real numbers satisfying*

$$(4.2) \quad \sum_{i=1}^{k-1} \ell_i = n,$$

for some positive real number n . Then

(a) If $k = 2$, we have

$$\int_{I_2} (\lambda_1 - \lambda_2)^n d\lambda^{(2)} = \frac{1}{(1+n)(2+n)}.$$

(b) If $k \geq 3$, we obtain

$$\int_{I_k} \prod_{i=1}^{k-1} (\lambda_i - \lambda_{i+1})^{\ell_i} d\lambda^{(k)} = \frac{1}{(k+n)(\ell_{k-1}+1)} \prod_{i=1}^{k-2} \int_0^1 (1-s)^{\ell_i} s^{k+n-(i+1)+\sum_{j=1}^i \ell_j} ds.$$

Proof. We only need to prove (b). Let $\lambda_1 \in (0, 1)$ be fixed. Consider the following change of variables

$$\lambda_{i+1} = \lambda_i s_i,$$

for $i \in \{1, \dots, k-1\}$. Using the fact that $0 < \lambda_{i+1} < \lambda_i$ we must have that $s_i \in (0, 1)$. Notice that this change of variables yields

$$(4.3) \quad \lambda_{i+1} = \lambda_1 \prod_{j=1}^i s_j.$$

Thus, the Jacobian associated to this change of variables is the determinant of an upper triangular matrix and it is given explicitly by the following formula.

$$\frac{\partial(\lambda_2, \dots, \lambda_k)}{\partial(s_1, \dots, s_{k-1})} = \lambda_1^{k-1} \prod_{i=1}^{k-2} s_i^{k-(i+1)}.$$

Observe that by (4.3) and (4.2) we have

$$\prod_{i=2}^{k-1} \lambda_i^{\ell_i} = \prod_{i=2}^{k-1} \left(\lambda_1 \prod_{j=1}^{i-1} s_j \right)^{\ell_i} = \lambda_1^{\sum_{j=2}^{k-1} \ell_j} \left(\prod_{i=1}^{k-2} s_i^{\sum_{j=i+1}^{k-1} \ell_j} \right) = \lambda_1^{n-\ell_1} \prod_{i=1}^{k-2} s_i^{n-\sum_{j=1}^i \ell_j}.$$

From this we conclude that

$$\begin{aligned} \prod_{i=1}^{k-1} (\lambda_i - \lambda_{i+1})^{\ell_i} &= \prod_{i=1}^{k-1} \lambda_i^{\ell_i} (1 - s_i)^{\ell_i} = \lambda_1^{\ell_1} (1 - s_{k-1})^{\ell_{k-1}} \prod_{i=1}^{k-2} (1 - s_i)^{\ell_i} \prod_{i=2}^{k-1} \lambda_i^{\ell_i} \\ &= \lambda_1^n (1 - s_{k-1})^{\ell_{k-1}} \prod_{i=1}^{k-2} (1 - s_i)^{\ell_i} \prod_{i=1}^{k-2} s_i^{n-\sum_{j=1}^i \ell_j}. \end{aligned}$$

As a result, integrating both sides of the above identity, we see that (b) is a consequence of the following equality.

$$\begin{aligned} \int_{I_k} \prod_{i=1}^{k-1} (\lambda_i - \lambda_{i+1})^{\ell_i} d\lambda^{(k)} &= \int_0^1 \lambda_1^{n+k-1} d\lambda_1 \int_0^1 (1 - s_{k-1})^{\ell_{k-1}} ds_{k-1} \\ &\quad \times \int_{[0,1]^{k-2}} \prod_{i=1}^{k-2} (1 - s_i)^{\ell_i} s_i^{n+k-(i+1)+\sum_{j=1}^i \ell_j} ds^{(k-2)}. \end{aligned}$$

□

For the computations to be performed below is worth recalling that

$$\mathcal{E}_\alpha(V) = \int_{\mathbb{R}^d} (-\Delta)^{\frac{\alpha}{2}} V(\theta) V(\theta) d\theta = \int_{\mathbb{R}^d} \widehat{V}(-\theta) \widehat{V}(\theta) |\theta_1|^\alpha \bar{d}\theta = \int_{\mathbb{R}^d} |\widehat{V}(\theta)|^2 |\theta|^\alpha \bar{d}\theta.$$

Lemma 4.2.

$$C_3(V) = \frac{1}{3!} \left(\int_{\mathbb{R}^d} V^3(\theta) d\theta + \mathcal{E}_\alpha(V) \right).$$

Proof. By Theorem 3.1, we have

$$C_3(V) = C_{0,3}(V) + C_{1,2}(V).$$

From (4.1), it suffices to compute $C_{1,2}(V)$. Following Corollary 3.1, we have, by Plancherel Theorem and (1.2), that

$$C_{1,2}(V) = A(1, 1) \int_{\mathbb{R}^d} \widehat{V}(-\theta_1) \widehat{V}(\theta_1) |\theta_1|^\alpha \bar{d}\theta_1 = \frac{1}{6} \int_{\mathbb{R}^d} V(\theta) (-\Delta)^{\frac{\alpha}{2}} V(\theta) d\theta,$$

which gives the formula above. \square

Lemma 4.3.

$$C_4(V) = \frac{1}{4!} \left(\int_{\mathbb{R}^d} V^4(\theta) d\theta + 2 \int_{\mathbb{R}^d} V^2(\theta) (-\Delta)^{\frac{\alpha}{2}} V(\theta) d\theta + \int_{\mathbb{R}^d} |(-\Delta)^{\frac{\alpha}{2}} V(\theta)|^2 d\theta \right).$$

Proof. By Theorem 3.1,

$$C_4(V) = C_{0,4}(V) + C_{1,3}(V) + \frac{1}{2!} C_{2,2}(V).$$

By Corollary 3.1 with $n = 1$ and $k = 3$, we have

$$\begin{aligned} C_{1,3}(V) = & A(1, (1, 0)) \int_{\mathbb{R}^{2d}} \widehat{V}(\theta_2) \widehat{V}(\theta_1) \widehat{V}(-\theta_1 - \theta_2) |\theta_1|^\alpha \bar{d}\theta_2 \bar{d}\theta_1 \\ & + A(1, (0, 1)) \int_{\mathbb{R}^{2d}} \widehat{V}(\theta_2) \widehat{V}(\theta_1) \widehat{V}(-\theta_1 - \theta_2) |\theta_1 + \theta_2|^\alpha \bar{d}\theta_2 \bar{d}\theta_1. \end{aligned}$$

From Lemma 4.1, we obtain

$$\begin{aligned} A(1, (1, 0)) &= \binom{1}{1, 0} \frac{1}{4} \int_0^1 (1-s) s ds = \frac{1}{4!}, \\ A(1, (0, 1)) &= \binom{1}{0, 1} \frac{1}{4 \cdot 2} \int_0^1 s^2 ds = \frac{1}{4!}. \end{aligned}$$

On the other hand, due to the basic properties of the Fourier transform,

$$\begin{aligned} & \int_{\mathbb{R}^{2d}} \widehat{V}(\theta_2) \widehat{V}(\theta_1) \widehat{V}(-\theta_1 - \theta_2) |\theta_1 + \theta_2|^\alpha \bar{d}\theta_2 \bar{d}\theta_1 \\ &= \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} \widehat{V}(\theta_2) \widehat{(-\Delta)^{\frac{\alpha}{2}} V(-\theta_1 - \theta_2)} \bar{d}\theta_2 \right) \widehat{V}(\theta_1) \bar{d}\theta_1 \\ &= \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} e^{i\theta \cdot \theta_1} V(\theta) (-\Delta)^{\frac{\alpha}{2}} V(\theta) d\theta \right) \widehat{V}(\theta_1) \bar{d}\theta_1 \\ &= \int_{\mathbb{R}^d} V^2(\theta) (-\Delta)^{\frac{\alpha}{2}} V(\theta) d\theta. \end{aligned}$$

A similar argument yields

$$\begin{aligned}
& \int_{\mathbb{R}^{2d}} \widehat{V}(\theta_2) \widehat{V}(\theta_1) \widehat{V}(-\theta_1 - \theta_2) |\theta_1|^\alpha \bar{d}\theta_2 \bar{d}\theta_1 \\
&= \int_{\mathbb{R}^d} |\theta_1|^\alpha \widehat{V}(\theta_1) \left(\int_{\mathbb{R}^d} \widehat{V}(-\theta_1 - \theta_2) \widehat{V}(\theta_2) \bar{d}\theta_2 \right) \bar{d}\theta_1 \\
&= \int_{\mathbb{R}^d} |\theta_1|^\alpha \widehat{V}(\theta_1) \left(\int_{\mathbb{R}^d} e^{i\theta_1 \cdot \theta} V^2(\theta) d\theta \right) \bar{d}\theta_1 \\
&= \int_{\mathbb{R}^d} V^2(\theta) (-\Delta)^{\frac{\alpha}{2}} V(\theta) d\theta.
\end{aligned}$$

Thus, we arrive at

$$C_{1,3}(V) = \frac{2}{4!} \int_{\mathbb{R}^d} V^2(\theta) (-\Delta)^{\frac{\alpha}{2}} V(\theta) d\theta.$$

Next,

$$C_{2,2}(V) = A(2, 2) \int_{\mathbb{R}^d} \widehat{V}(-\theta_1) \widehat{V}(\theta_1) |\theta_1|^{2\alpha} \bar{d}\theta_1 = \frac{2!}{4!} \int_{\mathbb{R}^d} |(-\Delta)^{\frac{\alpha}{2}} V(\theta)|^2 d\theta.$$

Therefore, the announced formula for $C_4(V)$ follows from the above identities. \square

Lemma 4.4.

$$\begin{aligned}
C_5(V) = \frac{1}{5!} & \left(\int_{\mathbb{R}^d} V^5(\theta) d\theta + 2 \int_{\mathbb{R}^d} V^3(\theta) (-\Delta)^{\frac{\alpha}{2}} V(\theta) d\theta + 2 \int_{\mathbb{R}^d} V^2(\theta) (-\Delta)^{\frac{\alpha}{2}} V(\theta) d\theta \right. \\
& \left. + \int_{\mathbb{R}^d} V(\theta) |(-\Delta)^{\frac{\alpha}{2}} V(\theta)|^2 d\theta + \mathcal{E}_\alpha((-\Delta)^{\frac{\alpha}{2}} V) + \mathcal{E}_\alpha(V^2) \right),
\end{aligned}$$

where $(-\Delta)^{\frac{\alpha}{2}}$ denotes the composition of $(-\Delta)^{\frac{\alpha}{2}}$ with itself.

Proof. Once again, Theorem 3.1 gives

$$C_5(V) = C_{0,5}(V) + C_{1,4}(V) + \frac{1}{2!} C_{2,3}(V) + \frac{1}{3!} C_{3,2}(V).$$

The first term $C_{0,5}(V)$ follows from (4.1). From Corollary 3.1 with $n = 1$ and $k = 4$, we have

$$(4.4) \quad C_{1,4}(V) = A(1, (1, 0, 0)) \int_{\mathbb{R}^{3d}} \widehat{V}(-\sum_{i=1}^3 \theta_i) \prod_{i=1}^3 \widehat{V}(\theta_i) |\theta_1|^\alpha \bar{d}\theta^{(3)}$$

$$(4.5) \quad + A(1, (0, 1, 0)) \int_{\mathbb{R}^{3d}} \widehat{V}(-\sum_{i=1}^3 \theta_i) \prod_{i=1}^3 \widehat{V}(\theta_i) \left| \sum_{m=1}^2 \theta_m \right|^\alpha \bar{d}\theta^{(3)}$$

$$(4.6) \quad + A(1, (0, 0, 1)) \int_{\mathbb{R}^{3d}} \widehat{V}(-\sum_{i=1}^3 \theta_i) \prod_{i=1}^3 \widehat{V}(\theta_i) \left| \sum_{m=1}^3 \theta_m \right|^\alpha \bar{d}\theta^{(3)}.$$

The most difficult term to compute in the above equality is the one appearing in (4.5) and we proceed to deal with this one first. By integrating first with respect

to θ_3 and applying Plancherel formula gives

$$(4.7) \quad \int_{\mathbb{R}^{3d}} \widehat{V}\left(-\sum_{i=1}^3 \theta_i\right) \prod_{i=1}^3 \widehat{V}(\theta_i) \left| \sum_{m=1}^2 \theta_m \right|^\alpha \bar{d}\theta^{(3)} \\ = \int_{\mathbb{R}^d} V^2(\theta) \left(\int_{\mathbb{R}^{2d}} \widehat{V}(\theta_1) \widehat{V}(\theta_2) |\theta_1 + \theta_2|^\alpha e^{i\theta \cdot (\theta_1 + \theta_2)} \bar{d}\theta_1 \bar{d}\theta_2 \right) d\theta.$$

Consider the change of variable $\theta_2 = z - \theta_1$, where the independent variable is θ_2 . Then, the integral in (4.7) between parenthesis equals

$$\int_{\mathbb{R}^{2d}} \widehat{V}(\theta_1) \widehat{V}(z - \theta_1) |z|^\alpha e^{i\theta \cdot z} \bar{d}\theta_1 \bar{d}z.$$

Thus, integrating the last expression with respect to θ_1 gives

$$\int_{\mathbb{R}^d} |z|^\alpha e^{i\theta \cdot z} \left(\int_{\mathbb{R}^d} e^{-i\eta \cdot z} V^2(\eta) d\eta \right) \bar{d}z = \int_{\mathbb{R}^d} |z|^\alpha e^{i\theta \cdot z} \widehat{V^2}(z) \bar{d}z = (-\Delta)^{\frac{\alpha}{2}} V^2(\theta).$$

In other words, we have shown that

$$\int_{\mathbb{R}^{3d}} \widehat{V}\left(-\sum_{i=1}^3 \theta_i\right) \prod_{i=1}^3 \widehat{V}(\theta_i) \left| \sum_{m=1}^2 \theta_m \right|^\alpha \bar{d}\theta^{(3)} = \mathcal{E}_\alpha(V^2).$$

Next, we claim that the other two integral terms in (4.6) equal

$$\int_{\mathbb{R}^d} V^3(\theta) (-\Delta)^{\frac{\alpha}{2}} V(\theta) d\theta.$$

To see this, it suffices to consider the following equalities.

$$\int_{\mathbb{R}^{2d}} \left(\int_{\mathbb{R}^d} \widehat{V}(-\theta_1 - \theta_2 - \theta_3) |\theta_1 + \theta_2 + \theta_3|^\alpha \widehat{V}(\theta_3) \bar{d}\theta_3 \right) \widehat{V}(\theta_1) \widehat{V}(\theta_2) \bar{d}\theta_1 \bar{d}\theta_2 \\ = \int_{\mathbb{R}^{2d}} \left(\int_{\mathbb{R}^d} e^{i\theta \cdot (\theta_1 + \theta_2)} V(\theta) (-\Delta)^{\frac{\alpha}{2}} V(\theta) d\theta \right) \widehat{V}(\theta_1) \widehat{V}(\theta_2) \bar{d}\theta_1 \bar{d}\theta_2 \\ = \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^{2d}} e^{i\theta \cdot (\theta_1 + \theta_2)} \widehat{V}(\theta_1) \widehat{V}(\theta_2) \bar{d}\theta_1 \bar{d}\theta_2 \right) V(\theta) (-\Delta)^{\frac{\alpha}{2}} V(\theta) d\theta \\ = \int_{\mathbb{R}^d} V^3(\theta) (-\Delta)^{\frac{\alpha}{2}} V(\theta) d\theta,$$

and

$$\int_{\mathbb{R}^{2d}} \left(\int_{\mathbb{R}^d} \widehat{V}(-\theta_1 - \theta_2 - \theta_3) \widehat{V}(\theta_3) \bar{d}\theta_3 \right) |\theta_1|^\alpha \widehat{V}(\theta_1) \widehat{V}(\theta_2) \bar{d}\theta_1 \bar{d}\theta_2 \\ = \int_{\mathbb{R}^{2d}} \left(\int_{\mathbb{R}^d} e^{i\theta \cdot (\theta_1 + \theta_2)} V^2(\theta) d\theta \right) \widehat{(-\Delta)^{\frac{\alpha}{2}} V}(\theta_1) \widehat{V}(\theta_2) \bar{d}\theta_1 \bar{d}\theta_2 \\ = \int_{\mathbb{R}^d} V^3(\theta) (-\Delta)^{\frac{\alpha}{2}} V(\theta) d\theta.$$

As far for the quantities, $A(1, (1, 0, 0))$, $A(1, (0, 1, 0))$ and $A(1, (0, 0, 1))$ we have

$$\begin{aligned} A(1, (1, 0, 0)) &= \frac{1}{5} \int_0^1 (1-s)s^2 ds \int_0^1 s ds = \frac{1}{5!} \\ A(1, (0, 1, 0)) &= \frac{1}{5} \int_0^1 s^3 ds \int_0^1 (1-s)s ds = \frac{1}{5!} \\ A(1, (0, 0, 1)) &= \frac{1}{5} \cdot \frac{1}{2} \int_0^1 s^3 ds \int_0^1 s^2 ds = \frac{1}{5!}. \end{aligned}$$

Therefore, we conclude that

$$(4.8) \quad C_{1,4}(V) = \frac{1}{5!} \left(2 \int_{\mathbb{R}^d} V^3(\theta) (-\Delta)^{\frac{\alpha}{2}} V(\theta) d\theta + \mathcal{E}_\alpha(V^2) \right).$$

Next, we compute $C_{2,3}(V)$. This time we have

$$\begin{aligned} (4.9) \quad C_{2,3}(V) &= A(2, (1, 1)) \int_{\mathbb{R}^{2d}} \widehat{V}(-\theta_1 - \theta_2) \widehat{V}(\theta_1) \widehat{V}(\theta_2) |\theta_1|^\alpha |\theta_1 + \theta_2|^\alpha \bar{d}\theta^{(2)} \\ &\quad + A(2, (2, 0)) \int_{\mathbb{R}^{2d}} \widehat{V}(-\theta_1 - \theta_2) \widehat{V}(\theta_1) \widehat{V}(\theta_2) |\theta_1|^{2\alpha} \bar{d}\theta^{(2)} \\ &\quad + A(2, (0, 2)) \int_{\mathbb{R}^{2d}} \widehat{V}(-\theta_1 - \theta_2) \widehat{V}(\theta_1) \widehat{V}(\theta_2) |\theta_1 + \theta_2|^{2\alpha} \bar{d}\theta^{(2)}. \end{aligned}$$

The first integral term in the right hand side of above equality equals

$$\begin{aligned} &\int_{\mathbb{R}^d} |\theta_1|^\alpha \widehat{V}(\theta_1) \left(\int_{\mathbb{R}^d} \widehat{V}(-\theta_1 - \theta_2) |\theta_1 + \theta_2|^\alpha \widehat{V}(\theta_2) \bar{d}\theta_2 \right) \bar{d}\theta_1 \\ &= \int_{\mathbb{R}^d} |\theta_1|^\alpha \widehat{V}(\theta_1) \left(\int_{\mathbb{R}^d} e^{i\theta \cdot \theta_1} V(\theta) (-\Delta)^{\frac{\alpha}{2}} V(\theta) d\theta \right) \bar{d}\theta_1 \\ &= \int_{\mathbb{R}^d} V(\theta) \left| (-\Delta)^{\frac{\alpha}{2}} V(\theta) \right|^2 d\theta. \end{aligned}$$

As for the other two integral terms in (4.9), we claim they both equal

$$\int_{\mathbb{R}^d} V^2(\theta) (-\Delta)^{\frac{\alpha}{2}} V(\theta) d\theta,$$

since the third integral term equals

$$\begin{aligned} &\int_{\mathbb{R}^d} \widehat{V}(\theta_1) \left(\int_{\mathbb{R}^d} \widehat{V}(\theta_2) \widehat{V}(-\theta_1 - \theta_2) |\theta_1 + \theta_2|^{2\alpha} \bar{d}\theta_2 \right) \bar{d}\theta_1 \\ &= \int_{\mathbb{R}^d} \widehat{V}(\theta_1) \left(\int_{\mathbb{R}^d} e^{i\theta \cdot \theta_1} V(\theta) (-\Delta)^{\frac{\alpha}{2}} V(\theta) d\theta \right) \bar{d}\theta_1, \end{aligned}$$

whereas the second one equals

$$\begin{aligned} &\int_{\mathbb{R}^d} |\theta_1|^{2\alpha} \widehat{V}(\theta_1) \left(\int_{\mathbb{R}^d} \widehat{V}(\theta_2) \widehat{V}(-\theta_1 - \theta_2) \bar{d}\theta_2 \right) \bar{d}\theta_1 \\ &= \int_{\mathbb{R}^d} |\theta_1|^{2\alpha} \widehat{V}(\theta_1) \left(\int_{\mathbb{R}^d} e^{i\theta \cdot \theta_1} V^2(\theta) d\theta \right) \bar{d}\theta_1. \end{aligned}$$

As for the coefficients in front of the integral terms, we have

$$\begin{aligned} A(2, (1, 1)) &= \binom{2}{1, 1} \frac{1}{5 \cdot 2} \int_0^1 (1-s)s^2 ds = \frac{2}{5!} \\ A(2, (2, 0)) &= \binom{2}{2, 0} \frac{1}{5} \int_0^1 (1-s)^2 s ds = \frac{2}{5!} \\ A(2, (0, 2)) &= \binom{2}{0, 2} \frac{1}{5 \cdot 3} \int_0^1 s^3 ds = \frac{2}{5!}. \end{aligned}$$

Therefore,

$$(4.10) \quad C_{2,3}(V) = \frac{2!}{5!} \int_{\mathbb{R}^d} V(\theta) |(-\Delta)^{\frac{\alpha}{2}} V(\theta)|^2 d\theta + \frac{2 \cdot 2!}{5!} \int_{\mathbb{R}^d} V^2(\theta) (-\Delta)^{\frac{\alpha}{2}} V(\theta) d\theta.$$

Likewise, we obtain

$$\begin{aligned} (4.11) \quad C_{3,2}(V) &= A(3, 3) \int_{\mathbb{R}^d} \widehat{V}(-\theta_1) \widehat{V}(\theta_1) |\theta_1|^{3\alpha} \bar{d}\theta_1 \\ &= \frac{1}{20} \int_{\mathbb{R}^d} (-\Delta)^{\frac{\alpha}{2}} V(\theta) (-\Delta)^{\frac{\alpha}{2}} V(\theta) d\theta = \frac{3!}{5!} \mathcal{E}_\alpha \left((-\Delta)^{\frac{\alpha}{2}} V \right). \end{aligned}$$

Combining (4.1), (4.8), (4.10), and (4.11), we obtain our expression for $C_5(V)$. \square

In the case of the Laplacian $\alpha = 2$, the signs of the coefficients can be used to give information on the poles on the metomorphic extension of the resolvent of the operator H_V ; see for example [6, Theorem 4.1]. In particular, it is shown in [6] that the first five coefficients in the trace expansion are non-negative provided the potential is non-negative. Our computations above yield a similar result for the first five coefficients of the heat content. More precisely we have

Corollary 4.1. *Suppose $V \in \mathcal{S}(\mathbb{R}^d)$, $V \geq 0$. Then $C_\ell(V) \geq 0$, for $1 \leq \ell \leq 5$.*

Proof. With

$$C_1(V) = \int_{\mathbb{R}^d} V(\theta) d\theta, \quad C_2(V) = \frac{1}{2} \int_{\mathbb{R}^d} V^2(\theta) d\theta,$$

and

$$C_3(V) = \frac{1}{3!} \left(\int_{\mathbb{R}^d} V^3(\theta) d\theta + \mathcal{E}_\alpha(V) \right),$$

the assertion trivially holds for these coefficients.

We can rewrite the expression in Lemma 4.3 for $C_4(V)$ as

$$\begin{aligned} C_4(V) &= \frac{1}{4!} \int_{\mathbb{R}^d} \left(V^4(\theta) + 2V^2(\theta) (-\Delta)^{\frac{\alpha}{2}} V(\theta) + |(-\Delta)^{\frac{\alpha}{2}} V(\theta)|^2 \right) d\theta \\ &= \frac{1}{4!} \int_{\mathbb{R}^d} |V^2(\theta) + (-\Delta)^{\frac{\alpha}{2}} V(\theta)|^2 d\theta \end{aligned}$$

and this shows that $C_4(V) \geq 0$.

For $C_5(V)$, we re-group the expression given by Lemma 4.4 as follows.

$$\begin{aligned}
C_5(V) &= \frac{1}{5!} \int_{\mathbb{R}^d} V(\theta) \left(V^4(\theta) + 2V^2(\theta)(-\Delta)^{\frac{\alpha}{2}} V(\theta) + |(-\Delta)^{\frac{\alpha}{2}} V(\theta)|^2 \right) d\theta \\
&+ \frac{1}{5!} \left(\mathcal{E}_\alpha \left((-\Delta)^{\frac{\alpha}{2}} V \right) + 2 \int_{\mathbb{R}^d} V^2(\theta)(-\Delta)^{\frac{\alpha}{2}} V(\theta) d\theta + \mathcal{E}_\alpha(V^2) \right) \\
&= \frac{1}{5!} \int_{\mathbb{R}^d} V(\theta) |V^2(\theta) + (-\Delta)^{\frac{\alpha}{2}} V(\theta)|^2 d\theta \\
&+ \frac{1}{5!} \left(\mathcal{E}_\alpha \left((-\Delta)^{\frac{\alpha}{2}} V \right) + 2 \int_{\mathbb{R}^d} V^2(\theta)(-\Delta)^{\frac{\alpha}{2}} V(\theta) d\theta + \mathcal{E}_\alpha(V^2) \right).
\end{aligned}$$

If V is non-negative the first of the last two terms above is clearly nonnegative. We claim the last term is also non-negative. To show this, we use Plancherel's identity for the second term and write the Dirichlet form in terms of the Fourier transform. However, we need to be a little careful here since the Fourier transform of a real valued function may be complex valued. Below we write $Re(z)$ for the real part of the complex number z and use the fact that for real valued functions, $\widehat{V}(-\theta) = \overline{\widehat{V}(\theta)}$. We write

$$\begin{aligned}
2 \int_{\mathbb{R}^d} V^2(\theta)(-\Delta)^{\frac{\alpha}{2}} V(\theta) d\theta &= \int_{\mathbb{R}^d} \widehat{V^2}(-\theta) |\theta|^{2\alpha} \widehat{V}(\theta) \bar{d}\theta + \int_{\mathbb{R}^d} \widehat{V^2}(\theta) |\theta|^{2\alpha} \widehat{V}(-\theta) \bar{d}\theta \\
&= \int_{\mathbb{R}^d} \overline{\widehat{V^2}(\theta)} |\theta|^{2\alpha} \widehat{V}(\theta) \bar{d}\theta + \int_{\mathbb{R}^d} \widehat{V^2}(\theta) |\theta|^{2\alpha} \overline{\widehat{V}(\theta)} \bar{d}\theta \\
&= 2 \int_{\mathbb{R}^d} |\theta|^{2\alpha} Re \left(\widehat{V^2}(\theta) \widehat{V}(\theta) \right) \bar{d}\theta.
\end{aligned}$$

Similarly,

$$\mathcal{E}_\alpha(V^2) = \int_{\mathbb{R}^d} |\theta|^\alpha |\widehat{V^2}(\theta)|^2 \bar{d}\theta \quad \text{and} \quad \mathcal{E}_\alpha \left((-\Delta)^{\frac{\alpha}{2}} V \right) = \int_{\mathbb{R}^d} |\theta|^{3\alpha} |\widehat{V}(\theta)|^2 \bar{d}\theta.$$

Putting these identities together gives

$$\begin{aligned}
&\mathcal{E}_\alpha \left((-\Delta)^{\frac{\alpha}{2}} V \right) + 2 \int_{\mathbb{R}^d} V^2(\theta)(-\Delta)^{\frac{\alpha}{2}} V(\theta) d\theta + \mathcal{E}_\alpha(V^2) \\
&= \int_{\mathbb{R}^d} \left(|\theta|^{2\alpha} |\widehat{V}(\theta)|^2 + 2|\theta|^\alpha Re \left(\widehat{V^2}(\theta) \widehat{V}(\theta) \right) + |\widehat{V^2}(\theta)|^2 \right) |\theta|^\alpha \bar{d}\theta \\
&= \int_{\mathbb{R}^d} \left| |\theta|^\alpha \widehat{V}(\theta) + \widehat{V^2}(\theta) \right|^2 |\theta|^\alpha \bar{d}\theta.
\end{aligned}$$

This together with our previous estimate shows that $C_5(V) \geq 0$, for $V \geq 0$. \square

Remark 4.1. *It is interesting to observe that for all $V \in \mathcal{S}(\mathbb{R}^d)$ (regardless of the sign), $C_2(V)$ and $C_4(V)$ are nonnegative. Whether or not this pattern remains as we move up along the even integers is an interesting question. With some patience one may be able to test this for $C_6(V)$ and perhaps even $C_8(V)$ but the general term is not clear at all.*

The probabilistic and Fourier transform techniques of this paper have been used recently in [2] to prove the existence of decompositions for additive functionals for one dimensional Cauchy and relativistic Cauchy stable processes. For more on this line of work, we refer the interested reader to [2], [24] and references therein.

REFERENCES

- [1] L. Acuña Valverde, *Trace asymptotics for Fractional Schrödinger operators*. J. Funct. Anal. **266**, 514-559, (2014).
- [2] L. Acuña Valverde, *A decomposition for additive functional of Lévy processes*. <http://arxiv.org/abs/1309.0142>, (2013).
- [3] R. Bañuelos, T. Kulczycki, *Trace estimates for stable processes*. Prob.Theory Relat. Fields. **142**, 313-338, (2008).
- [4] R. Bañuelos, T. Kulczycki, B. Siudeja, *On the heat trace of symmetry stables processes on Lipschitz domains*. J. Funct. Anal. **257**, 3329-3352, (2009).
- [5] R. Bañuelos, S. Yildirim, *Heat trace of non-local operators*. J.London Math. Society. **87**(1), 304-318, (2013).
- [6] R. Bañuelos, A. Sá Barreto, *On the heat trace of Schrödinger operators*. Comm in Partial Differential equations. **20**, 2153-2164, (1995).
- [7] M. van den Berg, E. B. Dryden and T. Kappeler, *Isospectrality and heat content*. <http://arxiv.org/abs/1304.4030v1>, (2013).
- [8] M. van den Berg, P. Gilkey, *Heat content asymptotics of a Riemannian manifold with boundary*. J. Funct. Anal. **120**, 48-71, (1994).
- [9] M. van den Berg, P. Gilkey, K. Kirsten, V. A. Kozlov, *Heat content asymptotics for Riemannian manifolds with Zaremba boundary conditions*. Potential Analysis **26**, 225-254, (2007).
- [10] M. van den Berg, *On the trace of the difference of Schrödinger heat semigroups*. Proceedings of the Royal Society of Edinburg. **119A**, 169-175, (1991).
- [11] M. van den Berg, *On the asymptotics of the heat equation and bounds on traces associated with Dirichlet Laplacian*. J. Funct. Anal. **71**, 279-293, (1987).
- [12] M. van den Berg, J.F le Gall, *Mean curvature and heat equation*. Math Z. **215**, 437-464, (1994).
- [13] J.Bertoin. Levy Processes. *Cambridge Tracts in Mathematics*. 1st edition, (1996).
- [14] H. Donnelly, *Compactness of isospectral potentials*. Trans. Amer. Math. Soc. **357**(5), 1717-1730, (2005).
- [15] M. Fukushima, Y. Oshima and M Takeda. Dirichlet Forms and Symmetric Markov Processes. *De Gruyter*. 2nd edition, (2010).
- [16] F. Hiroshima, T. Ichinose, J. Lőrinczi, *Path Integral Representation for Schrödinger Operators with Bernstein Functions of the Laplacian*. Rev. Math. Phys. **24**, 1250013 [40 pages], (2012).
- [17] M. Kac, *Can you hear the shape of a drum?* Am. Math. Mon. **73**, 1-23, (1966).
- [18] K. Kaleta, J. Lőrinczi, *Fractional $P(\phi)_1$ -processes and Gibbs measures*. <http://arxiv.org/pdf/1011.2713>, (2011).
- [19] H.P. McKean and I.M. Singer, *Curvature and the eigenvalues of the Laplacian*. J. Differ. Geom. **1**, 43-69, (1967).
- [20] H.P. McKean and P. van Moerbeke, *The spectrum of Hill's equation*. Inventiones Math. **30**, 217-274, (1975).
- [21] R.B. Melrose, *Scattering theory and the trace of the wave group*. J. Func. Anal. **45**, 29-44, (1982).
- [22] R.B. Melrose, *Geometric Scattering Theory. Lecture Notes, Stanford University*, (1994).
- [23] S. Minakshisundaram, *Eigenfunctions on Riemannian manifolds*. J. Indian Math. Soc. **17**, 158-165, (1953).
- [24] D. Nualart and Fangjun Xu, *Limits laws for occupation times of stables processes*. <http://arxiv.org/abs/1305.0241>, (2013).
- [25] B. Simon, *Schrödinger semigroups*. Bulletin of the AMS. **7**, 447-526, (1982).
- [26] Tiberiu Trif, *Multiple Integrals of Symmetric Functions*. The American Mathematical Monthly. **104**, 605-608, (1997).

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